MODEL-INDEPENDENT RELATIONS FOR THE MAGNETIC PROPERTIES OF THE SKYRMION

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A relation is derived between the densities of isovector charge and magnetic dipole moment of the pionic skyrmion, which is largely independent of the properties of the lagrangian, and this is used to obtain relations for the mean-squared radii. The results apply to both linear and non-linear σ-models.

1. Introduction

There has been much interest recently in the Skyrme model [1], in which baryons are seen as topological solitons ("skyrmions") in the pion field, stabilised by a term quartic in first derivatives of the field, the so-called "Skyrme term". In their seminal papers on the semi-classically quantised skyrmion [2,3], Adkins, Nappi and Witten (ANW) noticed that certain relations held independently of the parameters of this model, and this letter sets out to derive such relations for a general pionic $SU_2 \times SU_2$ lagrangian, subject to reasonable requirements of symmetry. Two relations (eqs. (37) and (49) below) for the charge and magnetic mean-squared radii of a skyrmion, are derived in the general case, and the three relations of ANW are recovered only if the total lagrangian can be written in the form

$$L = -M + 2\lambda \dot{a}^2, \tag{1}$$

where $\dot{a}$ describes the rate of rotation of the skyrmion.

In the following arguments, isotopic indices $\{\alpha, \beta, \gamma\}$ are taken to range over $\{1, ..., 3\}$, the index μ ranges over the axes of Minkowski space, and isotopic indices $\{\rho, \sigma, \tau\}$ range over $\{0, ..., 3\}$ in E_4.

We assume that the field ϕ_α is given by

$$\phi_\alpha = s(r) R_{\alpha j} \hat{x}_j, \tag{2}$$

where

$$R_{\alpha j} = a_\rho T^\alpha_{\rho\sigma} \tilde{T}^i_{\sigma\tau} a_\tau \equiv a T^\alpha \tilde{T}^i a, \tag{3}$$

$$a_\rho = a_\rho(t) \tag{4}$$

and

$$a_\rho^2 = 1, \tag{5}$$

and that

$$\phi_0 = \phi_0(r)\,. \tag{6}$$

The quantities T^α and $\tilde{T}^i$ are given by

$$T^\alpha_{\rho\sigma} = -\epsilon_{\alpha\rho\sigma} + \delta_{\rho 0}\delta_{\sigma\alpha} - \delta_{\sigma 0}\delta_{\rho\alpha},$$

$$\tilde{T}^\alpha_{\rho\sigma} = -\epsilon_{\alpha\rho\sigma} - \delta_{\rho 0}\delta_{\sigma\alpha} + \delta_{\sigma 0}\delta_{\rho\alpha},$$

$$\epsilon_{\alpha\rho 0} = 0,$$

and their algebra is given by Skyrme [1].

Relations (2) and (6) are obeyed by the spherical hedgehog ansatz, commonly taken to approximate to the skyrmion [2,1], but do not require the usual relation for the non-linear σ-model

$$\phi_\rho^2 = 1, \tag{7}$$

and the reasoning to be presented here can easily be extended to an n-component field.

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2. Charges

In terms of the four-component field ϕ_ρ, the density of angular momentum can be written

$$J_i^0 = \epsilon_{ijk} \frac{\partial \mathscr{L}}{\partial \dot{\phi}_\rho} \partial_j \phi_\rho x_k, \tag{8}$$

and in view of eqs. (2) and (6), this is

$$J_i^0 = \epsilon_{ijk} \frac{\partial \mathscr{L}}{\partial \dot{\phi}_\alpha} s(r) R_{\alpha j} \hat{x}_k. \tag{9}$$

Now consider the quantity p_ρ, which may be termed the density of conjugate momentum, defined by

$$p_\rho = \frac{\partial \mathscr{L}}{\partial \dot{a}_\rho}, \tag{10}$$

so that π_ρ, the conjugate momentum of $\dot{a}_\rho$, is

$$\pi_\rho = \int p_\rho \, \mathrm{d}^3 x. \tag{11}$$

Eqs. (2) and (6) give

$$p_\rho = 2 \frac{\partial \mathscr{L}}{\partial \dot{\phi}_\alpha} s(r) (T^\alpha \tilde{T}^j a)_\rho \hat{x}_j, \tag{12}$$

so that

$$a \tilde{T}^i p = -2 \frac{\partial \mathscr{L}}{\partial \dot{\phi}_\alpha} s(r) \epsilon_{ijk} R_{\alpha j} \hat{x}_k \tag{13}$$

or

$$J_i^0 = -\tfrac{1}{2} a \tilde{T}^i p, \tag{14}$$

and the total angular momentum is

$$J_i = -\tfrac{1}{2} a \tilde{T}^i \pi, \tag{15}$$

which gives the standard relation

$$\hat{J}_i = \tfrac{1}{2} \mathrm{i} a_\rho \tilde{T}^i_{\rho\sigma} \frac{\partial}{\partial a_\sigma}, \tag{16}$$

when the model is semi-classically quantized.

Turning to the isovector current, we can write

$$V_\alpha^\mu = -\epsilon_{\alpha\beta\gamma} \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_\beta)} \phi_\gamma, \tag{17}$$

which is normalised so that its charge I_α is the conventional hadronic isospin,

$$[I_\alpha, I_\beta] = \mathrm{i} \epsilon_{\alpha\beta\gamma} I_\gamma. \tag{18}$$

(This V_α^μ differs from that used by ANW by a factor of $\frac{1}{2}$.) Similar reasoning to that used for the angular momentum then yields

$$V_\alpha^0 = -\tfrac{1}{2} a T^\alpha p \tag{19}$$

and

$$I_\alpha = \int V_\alpha^0 \, \mathrm{d}^3 x = -\tfrac{1}{2} a T^\alpha \pi. \tag{20}$$

For simplicity, the classical quantities $-aT^\alpha \pi$ and $-a\tilde{T}^i \pi$ will be written as τ_α and σ_i.

3. The density of the magnetic moment

In terms of V_α^i, we can define an isovector magnetic moment

$$\mu_{\alpha i} = \int \rho_{\alpha i} \, \mathrm{d}^3 x, \tag{21}$$

where

$$\rho_{\alpha i} = \epsilon_{ijk} x_j V_\alpha^k \tag{22}$$

$$= \epsilon_{ijk} \epsilon_{lmn} R_{\alpha m} R_{\beta n} \frac{\partial \mathscr{L}}{\partial(\partial_j \phi_\beta)} r s(r) \hat{x}_k \hat{x}_l, \tag{23}$$

since

$$\epsilon_{\alpha\beta\gamma} R_{\gamma l} = \epsilon_{lmn} R_{\alpha m} R_{\beta n}. \tag{24}$$

The component μ_{3i} is the conventional isovector magnetic moment.

It will prove convenient to argue in terms of scalar quantities, so let us contract this with τ_α:

$$\tau_\alpha \rho_{\alpha i} = \epsilon_{ijk} \epsilon_{lmn} \frac{\partial \mathscr{L}}{\partial(\partial_j \phi_\beta)} R_{\beta n} (a \tilde{T}^m \pi) r s(r) \hat{x}_k \hat{x}_l. \tag{25}$$

We must now assume that the lagrangian density is Lorentz-invariant at each point, so that under an infinitesimal Lorentz transformation with origin at any given point, we can write

$$\delta L = 0 = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_\rho)} \delta(\partial_\mu \phi_\rho) \tag{26}$$

or

$$\frac{\partial \mathscr{L}}{\partial(\partial_j \phi_\rho)} \dot{\phi}_\rho = -\frac{\partial \mathscr{L}}{\partial \dot{\phi}_\rho} \partial_j \phi_\rho \tag{27}$$

at that point. Multiplying by $\epsilon_{ijk} \hat{x}_k$, and substituting in from eqs. (2) and (6), we obtain

$$\epsilon_{ijk} \frac{\partial \mathcal{L}}{\partial(\partial_j \phi_\alpha)} s(r) \dot{R}_{\alpha l} \hat{x}_k \hat{x}_l$$

$$= -\epsilon_{ijk} \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha} \frac{s(r)}{r} R_{\alpha j} \hat{x}_k, \tag{28}$$

and applying this to eq. (9),

$$J_i^0 = -\epsilon_{ijk} \frac{\partial \mathcal{L}}{\partial(\partial_j \phi_\alpha)} rs(r) \dot{R}_{\alpha l} \hat{x}_k \hat{x}_l$$

$$= 2\epsilon_{ijk} \epsilon_{lmn} \frac{\partial \mathcal{L}}{\partial(\partial_j \phi_\beta)} R_{\beta n} (a \tilde{T}^m \dot{a}) rs(r) \hat{x}_k \hat{x}_l, \tag{29}$$

since

$$\dot{R}_{\alpha l} = -2\epsilon_{lmn} R_{\alpha n} (a \tilde{T}^m \dot{a}). \tag{30}$$

If we now assume that the lagrangian density is invariant under real and isovector rotations, it can be shown that the total lagrangian can be written as a function of $\dot{a}_\rho^2$:

$$L = L(\dot{a}_\rho^2), \tag{31}$$

so that $\dot{a}_\rho$ is parallel to π_ρ, and we can write

$$\dot{a}_\rho = f(\pi_\rho^2) \pi_\rho. \tag{32}$$

(It is not strictly necessary for $f(\pi_\rho^2)$ to be single-valued, but we shall assume that it is.)

Eq. (25) can now be compared with eq. (29) to obtain

$$\frac{J_i^0}{f(\pi_\rho^2)} = 2\tau_\alpha \rho_{\alpha i}. \tag{33}$$

Using eqs. (14), (15), (19) and (20), we can write

$$\sigma_i J_i^0 = \tau_\alpha V_\alpha^0 = \tfrac{1}{2} \pi_\rho p_\rho, \tag{34}$$

whence we derive

$$\tau_\alpha V_\alpha^0 = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha} s(r) (a T^\alpha \tilde{T}^j \pi) \hat{x}_j, \tag{35}$$

which can be used to obtain the isovector mean-squared charge radius for a given lagrangian, and

$$\frac{\tau_\alpha V_\alpha^0}{f(\pi_\rho^2)} = 2\tau_\alpha \sigma_i \rho_{\alpha i}, \tag{36}$$

which is the desired relation between the densities of isovector charge and isovector magnetic moment, and from which the relation

$$\langle r^2 \rangle_{\mathrm{I}} = \tfrac{5}{3} \langle r^2 \rangle_{\mathrm{M,I}} \tag{37}$$

between the isovector mean-squared charge radius and magnetic radius, is derived [4]. Experiment indicates that eq. (37) is about 15% in error [3].

4. Gyromagnetic ratios

Integrating eq. (36), we find

$$\frac{\tau_\alpha^2}{f(\pi_\rho^2)} = \frac{\pi_\rho^2}{f(\pi_\rho^2)} = 4\tau_\alpha \sigma_i \mu_{\alpha i}. \tag{38}$$

We must now assume that the classical relation (38) applies to the expectation of the observables over nucleonic states $|\mathrm{N}\rangle$. Taking the definition of the isovector gyromagnetic ratio g_{I}

$$\langle \mathrm{N}' | \mu_{3i} | \mathrm{N} \rangle = \frac{g_{\mathrm{I}}}{4M_{\mathrm{N}}} \langle \mathrm{N}' | \tau_3 \sigma_i | \mathrm{N} \rangle, \tag{39}$$

for a nucleon with mass M_{N}, as an isovector equation

$$\langle \mathrm{N}' | \mu_{\alpha i} | \mathrm{N} \rangle = \frac{g_{\mathrm{I}}}{4M_{\mathrm{N}}} \langle \mathrm{N}' | \tau_\alpha \sigma_i | \mathrm{N} \rangle, \tag{40}$$

we find

$$\langle \mathrm{N}' | \tau_\alpha \sigma_i \mu_{\alpha i} | \mathrm{N} \rangle = \sum_{\mathrm{X}} \langle \mathrm{N}' | \tau_\alpha \sigma_i | \mathrm{X} \rangle \langle \mathrm{X} | \mu_{\alpha i} | \mathrm{N} \rangle, \tag{41}$$

where the sum is over all spin and isospin states $|\mathrm{X}\rangle$,

$$\langle \mathrm{N}' | \tau_\alpha \sigma_i \mu_{\alpha i} | \mathrm{N} \rangle = \sum_{\mathrm{N}''} \langle \mathrm{N}' | \tau_\alpha \sigma_i | \mathrm{N}'' \rangle \langle \mathrm{N}'' | \mu_{\alpha i} | \mathrm{N} \rangle$$

$$= \frac{g_{\mathrm{I}}}{4M_{\mathrm{N}}} \sum_{\mathrm{N}''} \langle \mathrm{N}' | \tau_\alpha \sigma_i | \mathrm{N}'' \rangle \langle \mathrm{N}'' | \tau_\alpha \sigma_i | \mathrm{N} \rangle$$

$$= \frac{9g_{\mathrm{I}}}{4M_{\mathrm{N}}} \langle \mathrm{N}' | \mathrm{N} \rangle. \tag{42}$$

Thus eq. (38) gives

$$\frac{9g_I}{M_{\mathrm{N}}} \langle \mathrm{N}' | \mathrm{N} \rangle = \langle \mathrm{N}' | \frac{\pi_\rho^2}{f(\pi_\rho^2)} | \mathrm{N} \rangle, \tag{43}$$

so

$$g_{\mathrm{I}} = \frac{M_{\mathrm{N}}}{3f(3)}. \tag{44}$$

The expression for the baryonic or isoscalar magnetic moment is well known to be [2]

$$\mu_{\mathrm{B}i} = -\tfrac{2}{3} \langle r^2 \rangle_{\mathrm{B}} a \tilde{T}^i \dot{a}, \tag{45}$$

where $\langle r^2 \rangle_B$ is the baryonic (isoscalar) mean-squared charge radius, so that

$$\sigma_i \mu_{Bi} = \tfrac{2}{3} \langle r^2 \rangle_B \sigma_i^2 f(\pi_\rho^2), \tag{46}$$

and from

$$\langle N' | \mu_{Bi} | N \rangle = \frac{g_B}{4M_N} \langle N' | \sigma_i | N \rangle \tag{47}$$

we find

$$g_B = \tfrac{8}{3} M_N \langle r^2 \rangle_B f(3). \tag{48}$$

Multiplying eq. (44) by (48), we finally obtain

$$\langle r^2 \rangle_B = \frac{9 g_I g_B}{8 M_N^2}, \tag{49}$$

which is the product of the two "mass relations" of ANW,

$$g_B = \tfrac{4}{9} \langle r^2 \rangle_B M_N (M_\Delta - M_N), \tag{50}$$

$$g_I = \frac{2M_N}{M_\Delta - M_N}, \tag{51}$$

and gives $\langle r^2 \rangle_B^{1/2} = 0.91$ fm, some 25% greater than the experimental value of 0.72 fm.

5. Conclusion

Two relations, (37) and (49), have been found for the charge and magnetic mean-squared radii of a skyrmion, valid for a large class of pionic lagrangians. Let us itemise the assumptions used in their derivation:

(1) The lagrangian density must be invariant under (a) rotational, (b) isovector and (c) Lorentz transformations;

(2) The three components ϕ_α of the field must have the form given by eq. (2), satisfied by the spherical hedgehog;

(3) All other components of the field must be constant with respect to time, and invariant under rotations;

(4) The field is scalar or pseudoscalar, and the three components ϕ_α are an isovector; all other components are isoscalar or zero;

(5) The classical equations which are derived can be taken directly as holding for the corresponding quantum-mechanical operators.

It is *not* assumed that the lagrangian is in any sense chirally symmetric, that the model is a non-linear σ-model (and in particular no constraint apart from items (3) and (5) above is placed on other components of the field), and it is not assumed that the field is an extremum of the action.

It should also be noted that if we assume that

$$L = -M + 2\lambda \dot{a}_\rho^2 \tag{52}$$

is a good approximation to the total lagrangian, as in the Skyrme model considered by ANW, then

$$f(\pi_\rho^2) = \frac{1}{4\lambda}, \tag{53}$$

and ANW's original "mass relations" eqs. (50) and (51) are recovered from eqs. (44) and (48).

Eqs. (35) and (36) may be of use in calculating the pionic contribution to the charge and magnetic moment in models which include other mesonic fields.

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References

[1] T.H.R. Skyrme, Proc. R. Soc. A 260 (1961) 127.
[2] G. Adkins et al., Nucl. Phys. B 228 (1983) 552.
[3] G. Adkins and C.R. Nappi, Nucl. Phys. B 228 (1983) 567.
[4] U.-G. Meißner et al., Nucl. Phys. A 466 (1987) 685.